EJERCICIO ELECTRODINÁMICA CUÁNTICA **CAPÍTULO 9**

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Ejercicio 9. Diagonalizar el hamiltoniano QED (parte fermiónica).

El hamiltoniano fermiónico QED tiene el siguiente aspecto:

$$H = \int d^3x \Psi^\dagger(- \gamma^0\gamma^a\partial_a + \gamma^0 m)\Psi$$

donde $\Psi^\dagger$, H y Ψ son operadores (c = 1 y $\hbar = 1$).

Por otra parte:

$$\Psi = \sum_{r=1}^{2}\int(d^3p/(2\pi)^3)\cdot(1/(2E_{\mathbf{p}})^{1/2})\cdot(a_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$$

$$\Psi^\dagger = \sum_{s=1}^{2}\int(d^3q/(2\pi)^3)\cdot(1/(2E_{\mathbf{q}})^{1/2})\cdot(a_s^\dagger(\mathbf{q})u_s^\dagger(\mathbf{q})e^{iqx} + d_s(\mathbf{q})v_s^\dagger(\mathbf{q})e^{-iqx})$$

donde $a^\dagger$, a, $d^\dagger$ y d son operadores creación y destrucción de partículas y antipartículas, respectivamente.

Si tenemos en cuenta:

$$i\partial_0 = - \gamma^0\gamma^a\partial_a + \gamma^0 m$$

$$H = \int d^3x \Psi^\dagger(i\partial_0)\Psi$$

Calculamos:

$$(i\partial_0)\Psi = i\partial_0\sum_{r=1}^{2}\int\left(d^3p/(2\pi)^3\right)\cdot\left(1/(2E_{\mathbf{p}})^{1/2}\right)\cdot\left(a_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right) =$$

$$= \sum_{r=1}^{2}\int\left(d^3p/(2\pi)^3\right)\cdot\left(1/(2E_{\mathbf{p}})^{1/2}\right)\cdot E_{\mathbf{p}}\left(a_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} - d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right) \qquad (p_0 = E_{\mathbf{p}})$$

Sustituyendo en H:

$$H = \sum_s\sum_r\int d^3x\int\left(d^3q/(2\pi)^3\right)\cdot\int\left(d^3p/(2\pi)^3\right)\cdot\left(1/(4E_{\mathbf{q}}E_{\mathbf{p}})^{1/2}\right)\cdot E_{\mathbf{p}}\left(a_s^{\dagger}(\mathbf{q})u_s^{\dagger}(\mathbf{q})e^{iqx} + d_s(\mathbf{q})v_s^{\dagger}(\mathbf{q})e^{-iqx}\right)\left(a_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} - d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right)$$

Hacemos el producto:

$$\left(a_s^{\dagger}(\mathbf{q})u_s^{\dagger}(\mathbf{q})e^{iqx} + d_s(\mathbf{q})v_s^{\dagger}(\mathbf{q})e^{-iqx}\right)\left(a_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} - d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right) =$$

$$a_s^{\dagger}(\mathbf{q})u_s^{\dagger}(\mathbf{q})a_r(\mathbf{p})u_r(\mathbf{p})e^{i(q-p)x} - a_s^{\dagger}(\mathbf{q})u_s^{\dagger}(\mathbf{q})d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{i(q+p)x} + d_s(\mathbf{q})v_s^{\dagger}(\mathbf{q})a_r(\mathbf{p})u_r(\mathbf{p})e^{-i(q+p)x} - d_s(\mathbf{q})v_s^{\dagger}(\mathbf{q})d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{-i(q-p)x}$$

Si integramos d^3x:

$$\int d^3x e^{\pm i(\mathbf{q}\pm\mathbf{p})x} = (2\pi)^3\delta^3(\mathbf{q}\pm\mathbf{p})$$

y luego integramos d^3q, se nos cancelan las dos integrales, un $(2\pi)^3$, y el producto anterior nos queda:

$$a_s^{\dagger}(\mathbf{p})u_s^{\dagger}(\mathbf{p})a_r(\mathbf{p})u_r(\mathbf{p}) - a_s^{\dagger}(-\mathbf{p})u_s^{\dagger}(-\mathbf{p})d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})e^{i(2p0)x0} + d_s(-\mathbf{p})v_s^{\dagger}(-\mathbf{p})a_r(\mathbf{p})u_r(\mathbf{p})e^{-i(2p0)x0} - d_s(\mathbf{p})v_s^{\dagger}(\mathbf{p})d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})$$

Los operadores creación y destrucción no actúan sobre espinores, por lo que podemos conmutarlos:

$$a_s^{\dagger}(\mathbf{p})a_r(\mathbf{p})u_s^{\dagger}(\mathbf{p})u_r(\mathbf{p}) - a_s^{\dagger}(-\mathbf{p})d_r^{\dagger}(\mathbf{p})u_s^{\dagger}(-\mathbf{p})v_r(\mathbf{p})e^{i(2p0)x0} + d_s(-\mathbf{p})a_r(\mathbf{p})v_s^{\dagger}(-\mathbf{p})u_r(\mathbf{p})e^{-i(2p0)x0} - d_s(\mathbf{p})d_r^{\dagger}(\mathbf{p})v_s^{\dagger}(\mathbf{p})v_r(\mathbf{p})$$

Si tenemos en cuenta que:

$$u_s^\dagger(\mathbf{p})u_r(\mathbf{p}) = 2E_\mathbf{p}\delta_{sr} \qquad u_s^\dagger(-\,\mathbf{p})v_r(\mathbf{p}) = 0 \qquad v_s^\dagger(-\,\mathbf{p})u_r(\mathbf{p}) = 0 \qquad v_s^\dagger(\mathbf{p})v_r(\mathbf{p}) = 2E_\mathbf{p}\delta_{sr}$$

$$H = \sum_s\sum_r\int\left(d^3p/(2\pi)^3\right)\cdot E_\mathbf{p}\left(a_s^\dagger(\mathbf{p})a_r(\mathbf{p})\delta_{sr} - d_s(\mathbf{p})d_r^\dagger(\mathbf{p})\delta_{sr}\right)$$

$$H = \sum_r\int\left(d^3p/(2\pi)^3\right)\cdot E_\mathbf{p}\left(a_r^\dagger(\mathbf{p})a_r(\mathbf{p}) - d_r(\mathbf{p})d_r^\dagger(\mathbf{p})\right)$$